Tessellations on the Poincaré Half-Plane and Disk.

Introduction

- The hyperbolic plane was inspired by a search for a space with a constant negative curvature. In $\mathbb{R}^3$, the sphere is a familiar comparison with constant positive curvature.To more easily work in a hyperbolic space, we often work with two different models on the euclidean complex plane, $\mathbb{C}$. The first model, and the one in which we chose to focus most on, is the Poincaré Half-Plane model. The second, named for the same mathematician, the Poincaré Disc-Model. An interesting and useful property of the hyperbolic plane is its ability to be tessellated or tiled- covered completely by a repeated pattern of geometric shapes- by any convex polygon.
- Our motivation for this project was create a program that would be able to tile the plane for many different isometry groups of $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$. Specifically, we wanted to quickly compare points which despite close proximity, have wildly different tilings.
- The most common base case uses the generators from the projective special linear group $PSL(2, \mathbb{Z})$ of matrices with determinant one.

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  which we see acting on the point 2i in the upper-left graph of the middle column.
- Also we considered the generating set with one element, such that all elements in the generated subgroup are exponents of the generator. In this case, the tiling is composed by polygon includes

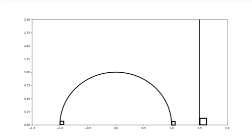
$$\exp = \begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$$

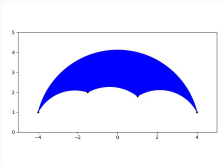
Background: definitions and previous results

Definition (Hyperbolic Length)
The *hyperbolic length* of a piece-wise smooth curve $\gamma : [a, b] \to \mathbb{H}$ is as usual the integral of the element of arc-length over the curve.

$$length_{\mathbb{H}}(\gamma) = \int_\gamma \frac{1}{\mathrm{Im}(z)} |dz| = \int_a^b \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))}$$

The distance between two points $a, b \in \mathbb{H}$, written $d_{\mathbb{H}}(a, b)$ is the length of the curve $\gamma_d \in \Gamma : [a, b] \to \mathbb{H} : length_{\mathbb{H}}(\gamma_d) = \mathrm{Inf}(length_{\mathbb{H}}(\Gamma)$; these curves are geodesics in the hyperbolic plane. The shortest path then manifests as segments of a straight vertical line or circle which intersects at 90° with the boundary.



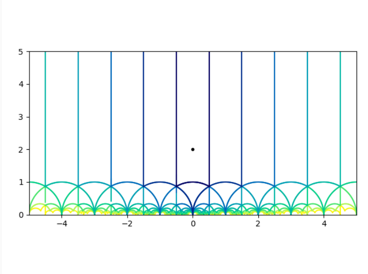


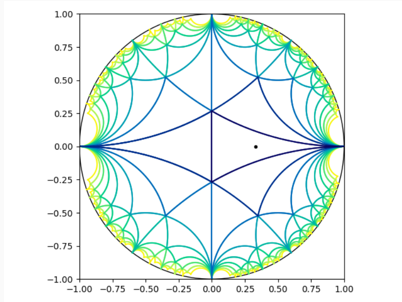
A convex polygon in $\mathbb{H}$. Let $P \in \mathbb{H}$ be a convex polygon. For any pair of points $x, y \in P$, the geodesic curve $\gamma_d([x, y])$ is entirely contained in P.

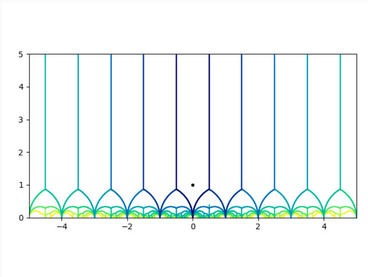
Definition (Möbius tranformation)
A *Möbius transformation*, described by the entries of a 2×2 abcd matrix- A, is a mapping $\phi : \mathbb{C} \to \mathbb{C}$ of the form

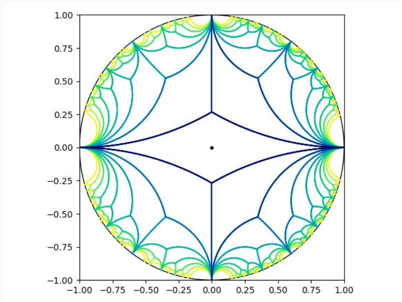
$$\phi_A(z) = \frac{az + b}{cz + d}$$

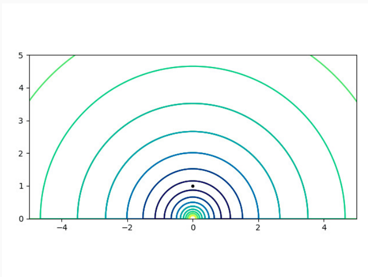
An example of a Möbius transformation which brings the disk to the half-plane $\phi : \mathbb{D} \to \mathbb{H}$ has entries $a, c = 1, b = -i, d = -1$

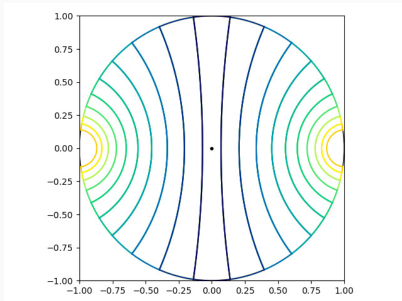












References

[1] Anderson, James W.. "Hyperbolic geometry." (1999).

[2] Klén, Riku Vuorinen, Matti. (2010). Apollonian circles and hyperbolic geometry.

[3] Soliman, A. A Beginner's Guide to Modular Curves.

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Fuchsian groups

Definition

- Since $\phi(-id) = \phi(id)$, we consider the quotient group

$$PSL(2, \mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\}$$

- A Fuchsian group is a discrete subgroup of PSL(2,R) = SL(2,R)/ $\pm id$, which can be regarded as a group of isometries of the hyperbolic plane.
- These are isometric homomorphisms- as in bijective and distance preserving.
- In $\mathbb{H}$, the set of $SL(2, \mathbb{Z})$, acting as Möbius transformations we write Möb($\mathbb{H}$), transforms a chosen point to generate a fundamental domain (polygon) which then tessellates the plane.

Fundamental Domain

Definition
We denote $\Gamma \in$ Möb($\mathbb{H}$) $= \phi_{SL(2,F)}$), a discrete subgroup. A *fundamental domain*, written as the open set U, for Γ on $\mathbb{H}$ is such that for $Id \in \Gamma$, Id the identity, $\phi_{Id}(U) = U$, and its closure $\overline{U}$ contains a fundamental set such that

$$\Gamma(\overline{U}) = \cup_{\gamma \in \Gamma} \gamma(\overline{U}) = \mathbb{H}$$

$$\text{If } g(\overline{U}) \cap h(\overline{U}) \neq \emptyset, \text{ then } g = h$$

An isometry of the hyperbolic plane is a mapping of the hyperbolic plane to itself that preserves the underlying hyperbolic geometry. The isometries of the hyperbolic plane form a group under composition.

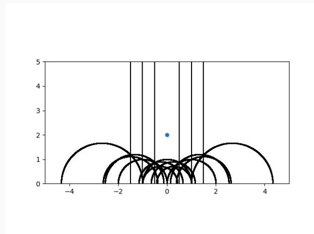
The Dirichlet Polygon

- The Dirichlet polygon can be defined as a set of points

$$F = \{z \in \mathbb{H} : d_{\mathbb{H}}(z, z_0) < d_{\mathbb{H}}(z, gz_0) : g \in \Gamma, g \neq Id\}$$

  where Γ is a fuschian group.
- The Dirichlet polygon is a fundamental domain.
- The Dirichlet polygon is convex in that the geodesic joining any two points of the polygon is contained entirely inside the polygon.

In the most common general case, we use the previously defined matrices U and V as well as the point $z_0 = 2i$. The fundamental domain (the set of points to be tiled) is a hyperbolic triangle with a vertex at infinity.



Side Pairing Transformation

Definition
For P, a hyperbolic polygon, any element $\gamma \in$ Möb($\mathbb{H}$) such that for sides s and s' of P, $\gamma(s' = s$. I.e. $\gamma(P) \cap P = s$.