Random Walks on Groups

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Introduction

Definition 1 *Let G be a group with a finite generating set S. The **Cayley graph of** G is a graph with vertex set given by the elements $g \in G$ and a directed edge corresponding to $s \in S$ between $g_1, g_2 \in G$ if $g_2 = g_1 \cdot s$. Using the length metric on the Cayley graph, we can define a metric on G, called the **word metric**. For more background, see [2].*

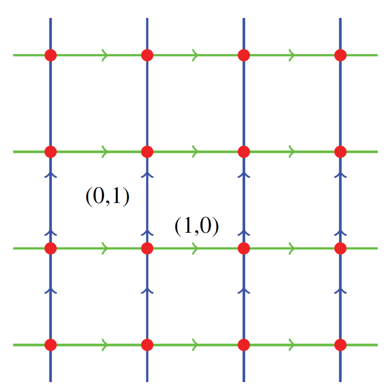

Fig. 1: Cayley graph of $\mathbb{Z}^2$ with respect to $\{\pm(1,0), \pm(0,1)\}$.

Definition 2 *A **nearest-neighbor random walk** on the Cayley graph of G is a random sequence of adjacent vertices. Usually, the first element is $id \in G$, and each successive element is determined by randomly choosing a generator from S and moving along that edge to get to the next vertex.*

Our goal is to study the long-term behavior of these random walks in infinite groups.

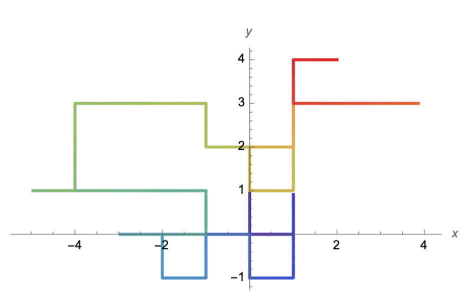

Fig. 2: Random walk on $\mathbb{Z}^2$ with respect to $\{\pm(1,0), \pm(0,1)\}$; action is component-wise addition

Random Walks in $\mathbb{Z}^2$

Using the word metric on $\mathbb{Z}^2$ associated to the standard generating set $\{\pm\{1,0\}, \pm\{0,1\}\}$, we get a metric related to the taxicab metric on $\mathbb{R}^2$.

Question 1 *How long will it take for a random walk in the Cayley graph of G to reach or hit a sphere of a given radius?*

Answer: In $\mathbb{Z}^2$, the expected number of steps it will take for the random walk to hit the taxicab sphere of a given radius grows quadrically with the radius. Moreover, the random walk is more likely to hit the sphere in the geometric center of each quadrant.

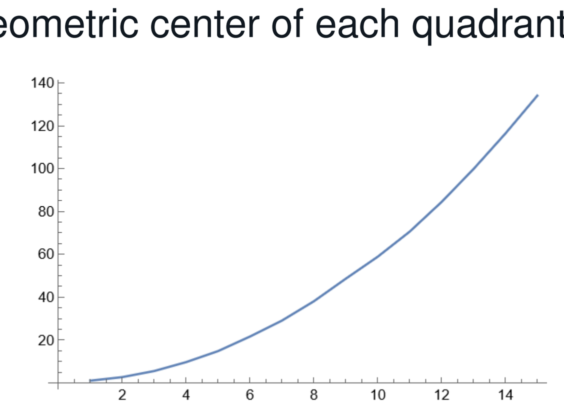

Fig. 3: Expected time to hit taxicab sphere of different radii (5000 trials)

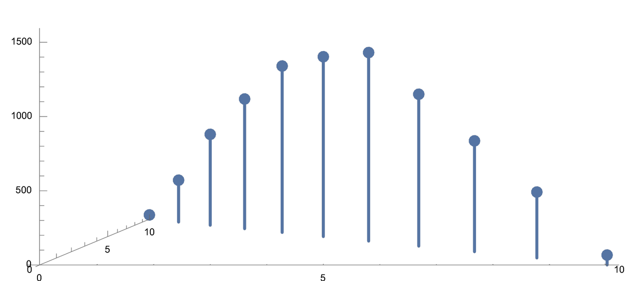

Fig. 4: Distribution of where the random walks hit sphere of radius 10

Random Walks in Heisenberg Group

Definition 3 *The **Heisenberg group** $H(\mathbb{R})$ is a 3-dimensional group that can be represented using exponential coordinates by elements $(x,y,z) \in \mathbb{R}^3$ and where the multiplication of elements is given by*

$$(x,y,z) \cdot (x',y',z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right).$$

Definition 4 *The **discrete Heisenberg group** $H(\mathbb{Z})$ is the set of lattice points in $H(\mathbb{R})$ generated by the **standard generating set** $S = \{\pm(1,0,0), \pm(0,1,0)\}$. Another generating set that we will investigate is $S_a = \{\pm(1,0,0), \pm(0,1,0) \pm (1,a,0)\}$ for small values of $a \in \mathbb{N}$.*

For any finite generating set of $H(\mathbb{Z})$, we can investigate random walks in the Cayley graph of $H(\mathbb{Z})$ equipped with the word metric. For the standard generating set S, we were able to implement the word metric given in [1]. To each word metric in $H(\mathbb{Z})$, there is also an associated sub-Finsler metric in $H(\mathbb{R})$ seeing the large-scale geometry [3]. The unit sphere in $H(\mathbb{R})$ of the sub-Finsler metric associated with the standard generators S, given in Figure 6, has four vertical walls (missing from the figure), a ceiling, and a basement. Using this sphere and a family of dilations in the group, we can partition $H(\mathbb{R})$ into vertical and horizontal regions to study the long-term behavior of random walks.

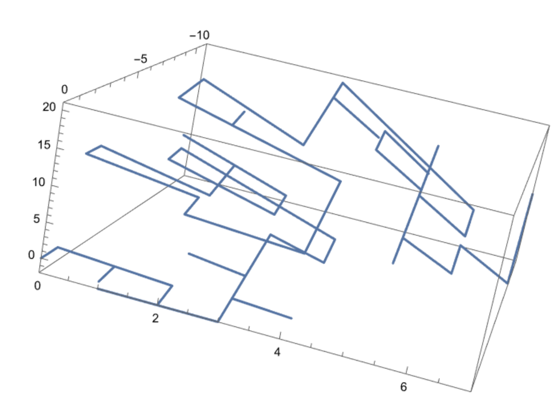

Fig. 5: Random walk in Cayley graph of $H(\mathbb{Z})$ with respect to $S = (\pm 1, 0, 0), (0, \pm 1, 0)$.

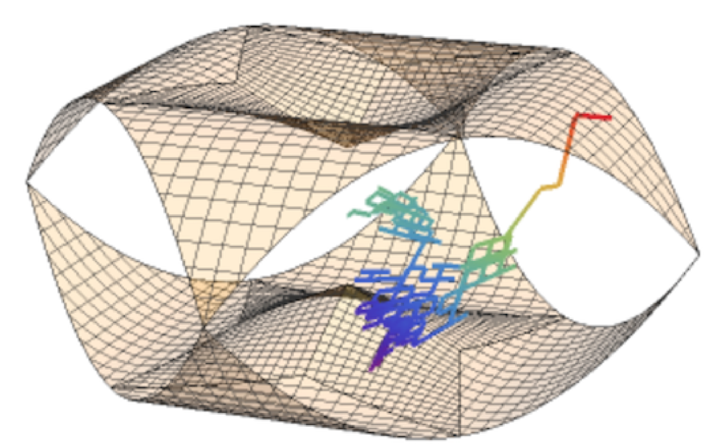

Fig. 6: Random walk hitting sphere of radius 20 in $H(\mathbb{R})$ with respect to $S = (\pm 1, 0, 0), (0, \pm 1, 0)$

Questions and Results

Question 2 *How does the average distance from the origin of a random walk in $H(\mathbb{Z})$, measured with a Heisenberg distance function, vary as the number of steps in the walk increases? And how does this answer change if we use different generating sets?*

Answer: Figure 7 shows the average distance of random walks of length $n \in \{1, \ldots, 100\}$ using generators S and S_1 and 100 trials for each n.

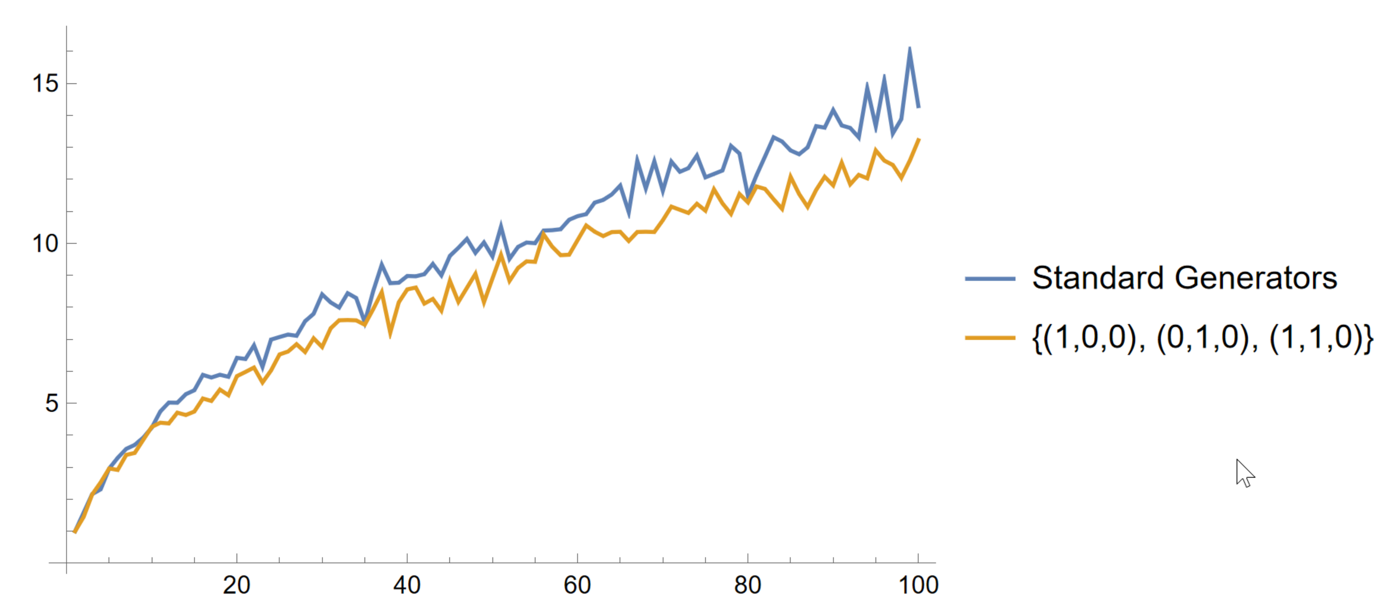

Fig. 7: Average distance from origin of random walk in $H(\mathbb{Z})$ with n steps

Questions and Results Cont.

Question 3 *For a given radius R and generating set S_a, how many steps should we expect to take before hitting the sphere of radius R in $H(\mathbb{R})$?*

Answer:

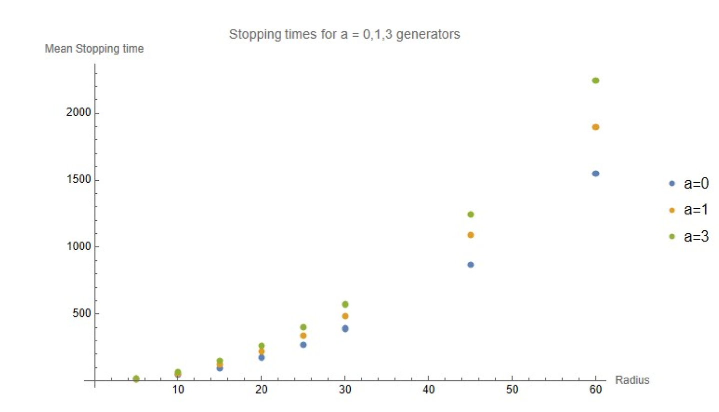

Fig. 8: Average number of steps to hit sphere of radius R for different values of a and R.

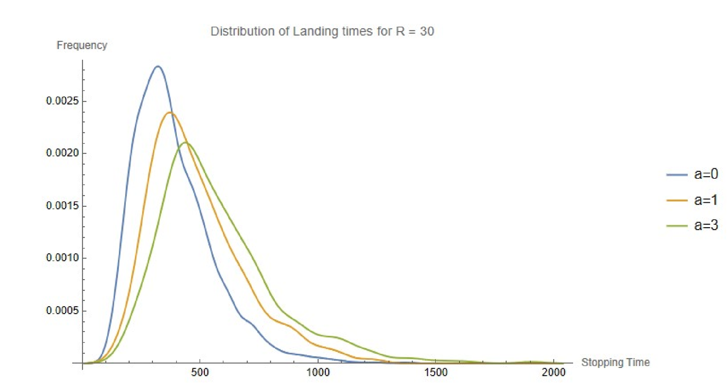

Fig. 9: Comparison of the distributions for hitting the sphere of radius $R = 30$ for $a = 0, 1, 3$.

Question 4 *Where on the sphere of radius R is a random walk more likely to hit? Is it more likely to be in a vertical wall or in a ceiling/basement region? How is this affected by the choice of generating set?*

Answer:

Generating Set	Vertical Wall	Ceiling/Basement
S	~0.4	~0.6
S_1	~0.75	~0.25
S_3	~0.85	~0.15
$S_a, a \geq 4$	~ 0.85	~ 0.15

Further Research

- Explore biased random walks on Heisenberg group with same questions
- Explore same questions with more generating sets of Heisenberg group to generalize current findings
- Explore the exact hitting region (instead of vertical/horizontal regions) of random walks at various radii, along with sequences that lead to it
- Explore similar topics on other groups, such as Lie groups

Acknowledgements

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References

[1] Sébastien Blachère. "Word distance on the discrete Heisenberg group". eng. In: *Colloquium Mathematicae* 95.1 (2003), pp. 21–36. URL: http://eudml.org/doc/285003.

[2] Matt Clay and Dan Margalit. *Office hours with a geometric group theorist.* Princeton University Press, 2017.

[3] Pierre Pansu. "Croissance des boules et des géodésiques fermées dans les nilvariétés". In: *Ergodic Theory Dynam. Systems* 3.3 (1983), pp. 415–445. ISSN: 0143-3857. DOI: 10.1017/S0143385700002054. URL: https://doi.org/10.1017/S0143385700002054.